

An upper bound for the number of perfect matchings in graphs

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Abstract

We give an upper bound on the number of perfect matchings in an undirected simple graph G with an even number of vertices, in terms of the degrees of all the vertices in G . This bound is sharp if G is a union of complete bipartite graphs. This bound is a generalization of the upper bound on the number of perfect matchings in bipartite graphs on $n+n$ vertices given by the Bregman-Minc inequality for the permanents of $(0,1)$ matrices.

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1 Introduction

Let $G = (V, E)$ be an undirected simple graph with the set of vertices V and edges E . For a vertex $v \in V$ denote by $\deg v$ the degree of the vertex v . Assume that $\#V$ is even. Denote by $\text{permat } G$ the number of perfect matching in G . Our main result states that

$$\text{permat } G \leq \prod_{v \in V} ((\deg v)!)^{\frac{1}{2 \deg v}}, \quad (1.1)$$

We assume here that $0^{\frac{1}{2}} = 0$. This result is sharp if G is a disjoint union of complete bipartite graphs. For bipartite graphs the above inequality follows from the Bregman-Minc inequality for the permanents of $(0,1)$ matrices, conjectured by Minc [4] and proved by Bregman [2]. In fact, the inequality (1.1) is the analog of the Bregman-Minc inequality for the *hafnians* of $(0,1)$ symmetric of even order with zero diagonal. Our proof follows closely the proof of the Bregman-Minc inequality given by Schrijver [6].

2 Permanents and Hafnians

If G is a bipartite graph on $n+n$ vertices then $\text{permat } G = \text{perm } B(G)$, where $B(G) = [b_{ij}] \in \{0,1\}^{n \times n}$ is the incidence matrix of the bipartite graph G . Thus $V = V_1 \cup V_2$ and $E \subset V_1 \times V_2$, where $V_i = \{v_{1,i}, \dots, v_{n,i}\}$ for $i = 1, 2$. Then $b_{ij} = 1$ if and only if $(v_{i,1}, v_{j,2}) \in E$. Recall that the permanent of $B \in \mathbb{R}^{n \times n}$ is given by $\text{perm } B = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n b_{i\sigma(i)}$, where \mathcal{S}_n is the symmetric group of all permutations $\sigma : \langle n \rangle \rightarrow \langle n \rangle$.

Vice versa, given any $(0,1)$ matrix $B = [a_{ij}] \in \{0,1\}^{n \times n}$, then B is the incidence matrix of the induced $G(B) = (V_1 \cup V_2, E)$. Denote by $\langle n \rangle := \{1, \dots, n\}$, $m + \langle n \rangle := \{m+1, \dots, m+n\}$.

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$n\}$ for any two positive integers m, n . It is convenient to identify $V_1 = \langle n \rangle, V_2 = n + \langle n \rangle$. Then $r_i := \sum_{j=1}^n b_{ij}$ is the i -th degree of $i \in \langle n \rangle$. The celebrated Bregman-Minc inequality, conjectured by Minc [4] and proved by Bregman [2], states

$$\text{perm } B \leq \prod_{i=1}^n (r_i!)^{\frac{1}{r_i}}. \quad (2.1)$$

A simple proof Bregman-Minc inequality is given [6]. Furthermore the above inequality is generalized to nonnegative matrices. See [1, 5] for additional proofs of (2.1).

Proposition 2.1 *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $\#V_1 = \#V_2$. Then (1.1) holds. If G is a union of complete bipartite graphs then equality holds in (1.1).*

PROOF Assume that $\#V_1 = \#V_2 = n$. Clearly,

$$\text{perfm} G = \text{perm } B(G) = \text{perm } B(G)^\top = \sqrt{\text{perm } B(G)} \sqrt{\text{perm } B(G)^\top}.$$

Note that the i -th row sum of $B(G)^\top$ is the degree of the vertex $n + i \in V_2$. Apply the Bregman-Minc inequality to $\text{perm } B(G)$ and $\text{perm } B(G)^\top$ to deduce (1.1).

Assume that G is the complete bipartite graph $K_{r,r}$ on $r + r$ vertices. Then $B(K_{r,r}) = J_r = \{1\}^{r \times r}$. So $\text{perfm} K_{r,r} = r!$. Hence equality holds in (1.1). Assume that G is a (disjoint) union of G_1, \dots, G_L . Since $\text{perfm} G = \prod_{i=1}^L \text{perfm} G_i$, we deduce (1.1) is sharp if each G_i is a complete bipartite graph. ■

Let $A(G) \in \{0, 1\}^{m \times m}$ be the adjacency matrix of an undirected simple graph G on m vertices. Note that $A(G)$ is a symmetric matrix with zero diagonal. Vice versa, any symmetric $(0, 1)$ matrix with zero diagonal induces an undirected simple graph $G(A) = (V, E)$ on m vertices. Identify V with $\langle m \rangle$. Then r_i , the i -th row sum of A , is the degree of the vertex $i \in \langle m \rangle$.

Let K_{2n} be the complete graph on $2n$ vertices, and denote by $\mathcal{M}(K_{2n})$ the set of all perfect matches in K_{2n} . Then $\alpha \in \mathcal{M}(K_{2n})$ can be represented as $\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$ with $i_k < j_k$ for $k \in \langle n \rangle$. It is convenient to view (i_k, j_k) as an edge in K_{2n} . We can view α as an involution in S_{2n} with no fixed points. So for $l \in \langle 2n \rangle$ $\alpha(l)$ is second vertex corresponding to l in the perfect match given by α . Vice versa, any fixed point free involution of $\langle 2n \rangle$ induces a perfect match $\alpha \in \mathcal{M}(K_{2n})$. Denote by S_m the space of $m \times m$ real symmetric matrices. Assume that $A = [a_{ij}] \in S_{2n}$. Then the *hafnian* of A is defined as

$$\text{hafn } A := \sum_{\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\} \in \mathcal{M}(K_{2n})} \prod_{k=1}^n a_{i_k j_k}. \quad (2.2)$$

Note that $\text{hafn } A$ does not depend on the diagonal entries of A . Let $i \neq j \in \langle 2n \rangle$. Denote by $A(i, j) \in S_{2n-2}$ the symmetric matrix obtained from A by deleting the i, j rows and columns of A . The following proposition is straightforward, and is known as the expansion of the hafnian by the row, (column), i .

Proposition 2.2 *Let $A \in S_{2n}$. Then for each $i \in \langle 2n \rangle$*

$$\text{hafn } A = \sum_{j \in \langle 2n \rangle \setminus \{i\}} a_{ij} \text{hafn } A(i, j) \quad (2.3)$$

It is clear that $\text{perfm} G = \text{hafn } A(G)$ for any $G = (\langle 2n \rangle, E)$. Then (1.1) is equivalent to the inequality

$$\text{hafn } A \leq \prod_{i=1}^{2n} (r_i!)^{\frac{1}{2r_i}} \text{ for all } A \in \{0, 1\}^{(2n) \times (2n)} \cap S_{2n,0} \quad (2.4)$$

Our proof of the above inequality follows the proof of the Bregman-Minc inequality given by A. Schrijver [6].

3 Preliminaries

Recall that $x \log x$ is a strict convex function on $\mathbb{R}_+ = [0, \infty)$, where $0 \log 0 = 0$. Hence

$$\frac{\sum_{j=1}^r t_j}{r} \log \frac{\sum_{j=1}^r t_j}{r} \leq \frac{1}{r} \sum_{j=1}^r t_j \log t_j, \text{ for } t_1, \dots, t_r \in \mathbb{R}_+. \quad (3.1)$$

Clearly, the above inequality is equivalent to the inequality

$$\left(\sum_{j=1}^r t_j \right)^{\sum_{j=1}^r t_j} \leq r^{\sum_{j=1}^r t_j} \prod_{j=1}^r t_j^{t_j} \text{ for } t_1, \dots, t_r \in \mathbb{R}_+. \quad (3.2)$$

Here $0^0 = 1$.

Lemma 3.1 *Let $A = [a_{ij}] \in \{0, 1\}^{(2n) \times (2n)} \cap S_{2n,0}$. Then for each $i \in \langle 2n \rangle$*

$$(\text{hafn } A)^{\text{hafn } A} \leq r_i^{\text{hafn } A} \prod_{j, a_{ij}=1} (\text{hafn } A(i, j))^{\text{hafn } A(i, j)}. \quad (3.3)$$

PROOF Let $t_j = \text{hafn } A(i, j)$ for $a_{ij} = 1$. Use (2.3) and (3.2) to deduce (3.3). ■

To prove our main result we need the following two lemmas.

Lemma 3.2 *The sequence $(k!)^{\frac{1}{k}}, k = 1, \dots$, is an increasing sequence.*

PROOF Clearly, the inequality $(k!)^{\frac{1}{k}} < ((k+1)!)^{\frac{1}{k+1}}$ is equivalent to the inequality $(k!)^{k+1} < ((k+1)!)^k$, which is in turn equivalent to $k! < (k+1)^k$, which is obvious. ■

Lemma 3.3 *For an integer $r \geq 3$ the following inequality holds.*

$$(r!)^{\frac{1}{r}} ((r-2)!)^{\frac{1}{r-2}} < ((r-1)!)^{\frac{2}{r-1}}. \quad (3.4)$$

PROOF Raise the both sides of (3.4) to the power $r(r-1)(r-2)$ to deduce that (3.4) is equivalent to the inequality

$$(r!)^{(r-1)(r-2)} ((r-2)!)^{r(r-1)} < ((r-1)!)^{2r(r-2)}.$$

Use the identities

$$\begin{aligned} r! &= r(r-1)!, & (r-1)! &= (r-1)(r-2)!, \\ 2r(r-2) &= (r-1)(r-2) + r(r-1) - 2, & r(r-1) - 2 &= (r+1)(r-2) \end{aligned}$$

to deduce that the above inequality is equivalent to

$$r^{(r-1)(r-2)} ((r-2)!)^2 < (r-1)^{(r+1)(r-2)}.$$

Take the logarithm of the above inequality, divide it by $(r-2)$ deduce that (3.4) is equivalent to the inequality

$$(r-1) \log r + \frac{2}{r-2} \log(r-2)! - (r+1) \log(r-1) < 0.$$

This inequality is equivalent to

$$s_r := (r-1) \log \frac{r}{r-1} + 2 \left(\frac{1}{r-2} \log(r-2)! - \log(r-1) \right) < 0 \text{ for } r \geq 3. \quad (3.5)$$

Clearly

$$(r-1) \log \frac{r}{r-1} = (r-1) \log \left(1 + \frac{1}{r-1} \right) < (r-1) \frac{1}{r-1} = 1.$$

Hence (3.5) holds if

$$\frac{1}{r-2} \log(r-2)! - \log(r-1) < -\frac{1}{2}. \quad (3.6)$$

Recall the Stirling's formula [3, pp. 52]

$$\log k! = \frac{1}{2} \log(2\pi k) + k \log k - k + \frac{\theta_k}{12k} \text{ for some } \theta_k \in (0, 1). \quad (3.7)$$

Hence

$$\frac{\log(r-2)!}{r-2} < \frac{\log 2\pi(r-2)}{2(r-2)} + \log(r-2) - 1 + \frac{1}{12(r-2)^2}.$$

Thus

$$\frac{1}{r-2} \log(r-2)! - \log(r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} + \log \frac{r-2}{r-1} + \frac{1}{12(r-2)^2} - 1.$$

Since e^x is convex, it follows that $1+x \leq e^x$. Hence

$$\frac{1}{r-2} \log(r-2)! - \log(r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} - \frac{1}{r-1} + \frac{1}{12(r-2)^2} - 1.$$

Note that $-\frac{1}{r-1} + \frac{1}{12(r-2)^2} < 0$ for $r \geq 3$. Therefore

$$\frac{1}{r-2} \log(r-2)! - \log(r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} - 1. \quad (3.8)$$

Observe next that the function $\frac{\log 2\pi x}{2x}$ is decreasing for $x > \frac{e}{2\pi}$. Hence the right-hand side of (3.8) is a decreasing sequence for $r = 3, \dots$. Since $\frac{\log 2\pi \cdot 3}{2 \cdot 3} = 0.4894$, it follows that the right-hand side of (3.8) is less than -0.51 for $r \geq 5$. Therefore (3.5) holds for $r \geq 5$. Since

$$s_3 = \log \frac{9}{16} < 0, \quad s_4 = \log \frac{128}{243} < 0$$

we deduce the lemma. ■

The arguments of the Proof of Lemma 3.3 yield that $s_r, r = 3, \dots$, converges to -1 . We checked the values of this sequence for $r = 3, \dots, 100$, and we found that this sequence decreases in this range. We conjecture that the sequence $s_r, r = 3, \dots$ decreases.

4 Proof of generalized Bregman-Minc inequality

Theorem 4.1 *Let $G = (V, E)$ be undirected simple graph on an even number of vertices. Then the inequality (1.1) holds.*

PROOF We prove (2.4). We use the induction on n . For $n = 1$ (2.4) is trivial. Assume that theorem holds for $n = m - 1$. Let $n = m$. It is enough to assume that $\text{hafn } A > 0$. In particular each $r_i \geq 1$. If $r_i = 1$ for some i , then by expanding $\text{hafn } A$ by the row i , using the induction hypothesis and Lemma 3.2, we deduce easily the theorem in this case. Hence we assume that $r_i \geq 2$ for each $i \in \langle 2n \rangle$. Let $G = G(A) = (\langle 2n \rangle, E)$ be the graph induced by A . Then $\text{hafn } A > 0$ is the number of perfect matchings in G . Denote by $\mathcal{M} := \mathcal{M}(G) \subset \mathcal{M}(K_{2n})$ the set of all perfect matchings in G . Then $\#\mathcal{M} = \text{hafn } A$. We now follow the arguments in the proof of the Bregman-Minc theorem given in [6] with the corresponding modifications.

$$\begin{aligned}
(\text{hafn } A)^{2n} \text{hafn } A &\stackrel{(1)}{=} \prod_{i=1}^{2n} (\text{hafn } A)^{\text{hafn } A} \stackrel{(2)}{\leq} \prod_{i=1}^{2n} (r_i^{\text{hafn } A} \prod_{j, a_{ij}=1} (\text{hafn } A(i, j))^{\text{hafn } A(i, j)}) \\
&\stackrel{(3)}{=} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) (\prod_{i=1}^{2n} \text{hafn } A(i, \alpha(i)))) \\
&\stackrel{(4)}{\leq} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{i=1}^{2n} (\prod_{j \in \langle 2n \rangle \setminus \{i, \alpha(i)\}, a_{ij}=a_{\alpha(i)j}=0} (r_j!)^{\frac{1}{2r_j}})) \\
&(\prod_{j \in \langle 2n \rangle \setminus \{i, \alpha(i)\}, a_{ij}+a_{\alpha(i)j}=1} ((r_j - 1)!)^{\frac{1}{2(r_j-1)}}) (\prod_{j \in \langle 2n \rangle \setminus \{i, \alpha(i)\}, a_{ij}+a_{\alpha(i)j}=2} ((r_j - 2)!)^{\frac{1}{2(r_j-2)}})) \\
&\stackrel{(5)}{=} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} (\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}=a_{\alpha(i)j}=0} (r_j!)^{\frac{1}{2r_j}})) \\
&(\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}+a_{\alpha(i)j}=1} ((r_j - 1)!)^{\frac{1}{2(r_j-1)}}) (\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}+a_{\alpha(i)j}=2} ((r_j - 2)!)^{\frac{1}{2(r_j-2)}})) \\
&\stackrel{(6)}{\leq} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} ((r_j!)^{\frac{2n-2r_j}{2r_j}}) ((r_j - 1)!)^{\frac{2(r_j-1)}{2(r_j-1)}})) \\
&\stackrel{(7)}{=} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} (r_i!)^{\frac{2n}{2r_i}})) \stackrel{(8)}{=} (\prod_{i=1}^{2n} (r_i!)^{\frac{1}{2r_i}})^{2n} \text{hafn } A.
\end{aligned}$$

We now explain each step of the proof.

1. Trivial.
2. Use (3.3).
3. The number of factors of r_i is equal to $\text{hafn } A$ on both sides, while the number of factors $\text{hafn } A(i, j)$ equals to the number of $\alpha \in \mathcal{M}$ such that $\alpha(i) = j$.
4. Apply the induction hypothesis to each $\text{hafn } A(i, \alpha(i))$. Note that since the edge $(i, \alpha(i))$ appears in the perfect matching $\alpha \in \mathcal{M}$, it follows that $\text{hafn } A(i, \alpha(i)) \geq 1$. Hence if $j \in \langle 2n \rangle \setminus \{i, \alpha(i)\}$ and $r_j = 2$ we must have that $a_{ij} + a_{\alpha(i)j} \leq 1$.
5. Change the order of multiplication.
6. Fix $\alpha \in \mathcal{M}$ and $j \in \langle 2n \rangle$. Then j is matched with $\alpha(j)$. Consider all other $n - 1$ edges $(i, \alpha(i))$ in α . j is connected to $r_j - 1$ vertices in $\langle 2n \rangle \setminus \{j, \alpha(j)\}$. Assume there are s triangles formed by j and the s edges out of $n - 1$ edges in $\alpha \setminus (j, \alpha(j))$. Then j is connected to $t = r_j - 1 - 2s$ edges vertices $i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}$ such that j is not connected to $\alpha(i)$. Hence there are $2n - 2 - (2t + 2s)$ vertices $k \in \langle 2n \rangle \setminus \{j, \alpha(j)\}$ such that j is not connected to k and $\alpha(k)$. Therefore, for this α and j we have the following terms in (5):

$$\begin{aligned}
&(\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}=a_{\alpha(i)j}=0} (r_j!)^{\frac{1}{2r_j}}) (\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}+a_{\alpha(i)j}=1} ((r_j - 1)!)^{\frac{1}{2(r_j-1)}}) \\
&(\prod_{i \in \langle 2n \rangle \setminus \{j, \alpha(j)\}, a_{ij}+a_{\alpha(i)j}=2} ((r_j - 2)!)^{\frac{1}{2(r_j-2)}})) = \\
&(r_j!)^{\frac{2n-2-(2s+2t)}{2r_j}} ((r_j - 1)!)^{\frac{2t}{2(r_j-1)}} ((r_j - 2)!)^{\frac{2s}{2(r_j-2)}}) = \\
&(r_j!)^{\frac{2n-r_j-1}{2r_j}} ((r_j - 2)!)^{\frac{r_j-1}{2(r_j-2)!}} ((r_j!)^{-\frac{1}{r_j}} ((r_j - 2)!)^{-\frac{1}{(r_j-2)}} ((r_j - 1)!)^{\frac{2}{(r_j-1)}})^{\frac{t}{2}}. \quad (4.1)
\end{aligned}$$

In the last step we used the equality $r_j - 1 = 2s + t$. Assume first that $r_j > 2$. Use Lemma 3.3 to deduce that (4.1) increases in t . Hence the maximum value of (4.1) is achieved when $s = 0$ and $t = r_j - 1$. Then (4.1) is equal to

$$(r_j!)^{\frac{2n-2r_j}{2r_j}} ((r_j - 1)!)^{\frac{2(r_j-1)}{2(r_j-1)}}.$$

If $r_j = 2$ then, as we explained above, $s = 0$. Hence (4.1) is also equal to the above expression. Hence (6) holds.

7. Trivial.

8. Trivial.

Thus

$$(\text{hafn } A)^{2n \text{ hafn } A} \leq \left(\prod_{i=1}^{2n} (r_i!)^{\frac{1}{2r_i}} \right)^{2n \text{ hafn } A}.$$

This establishes (2.4). ■

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